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Rotational Invariance and the S Matrix in Nonrelativistic Quantum Mechanics*

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The connection between an observable invariance property of the scattering matrix and the interaction, V, generating the scattering, is studied for a particlar case: rotational invariance in nonrelativistic "potential" scattering. It is shown that a determination that the scattering cross section depends only on the angle between the incident and outgoing beams by no means implies that V is invariant under rotation. This is true even if coherent incident beams are used to probe the target. Some aspects of these results and their relevance for the construction of physical theories are briefly discussed.

1. INTRODUCTION

FROM the physical standpoint, it is interesting to ask what exactly is implied about the symmetry properties of an interaction if the observed scattering (due to this interaction) exhibits certain symmetries. Consider, for example, the case of elastic scattering in nonrelativistic quantum mechanics and the symmetry connected with the 3-dimensional rotation group. Generally, it has been taken or assumed that if the scattering is observed to be rotationally invariant then the interaction is rotationally symmetric. In classical physics, one might quite readily accept this hypothesis since there we are able, at least in principle, to make exact measurements at every instant and point in the experiment; and there we can talk of the "interaction" in terms of the "direct" fields, such as the electric field, which is tantamount to discussing the interaction in terms of the forces themselves. However, in quantum mechanics such a detailed description (in terms of physical measurements) is not possible.

In particular, suppose that the scattering, in the c.m. system, is governed by the time-independent Schrödinger equation,

$$(\mathbf{p}^2/2m+V)\psi_{\mathbf{k}}(\mathbf{x})=E_{\mathbf{k}}\psi_{\mathbf{k}}(\mathbf{x}).$$

Suppose that, for scattering states with incident plane waves, only the asymptotic form of the wave function is regarded as observable, i.e., $\psi_{\mathbf{k}}\sim e^{i\mathbf{k}\cdot\mathbf{x}}+f(\theta,\varphi)e^{ikr}/r$ and $|f|^2$ is determined by measuring a differential cross section. It is well known that such measurements by no means determine V—even if it is assumed that V is local. On the other hand, to the extent that $|\psi(\mathbf{x})|^2$ is regarded as measurable in the near zone, not just the radiation zone, V is correspondingly more closely determined. The point that we wish to make is that if, in accord with the S-matrix point of view initiated by Heisenberg, only $|f|^2$ is regarded as observable, the freedom in V may be so large that even a powerful symmetry property of $|f|^2$, corresponding to rotational invariance of the scattering, need not be shared by V—at least if V is not required to be local.

From the point of view of integro-differential equations, a rotationally noninvariant V can give rise to invariant $|f|^2$, and indeed f itself, by permitting a solution $\psi_{\mathbf{k}}(\mathbf{x})$, noninvariant in the near zone, where V is fully felt, to become asymptotically invariant in the sense

$$\psi_{\mathbf{k}}(\mathbf{x})-\psi_{R\mathbf{k}}(R\mathbf{x})=O(1/|\mathbf{x}|^2)\quad\text{as}\quad|\mathbf{x}|\to\infty.$$

Alternatively,

$$f=-\frac{m}{2\pi}\langle\mathbf{k}'|T|\mathbf{k}\rangle,$$

where T is the transition operator. From this point of view, the possibility of a noninvariant V giving rise to an invariant f corresponds to the fact that the matrix elements of the transition operator are taken only on the energy shell: $|\mathbf{k}'|=|\mathbf{k}|$. These two points of view are, of course, not unrelated.

In Sec. 2 some simple theorems relating mainly to local potentials are stated and proved. The results strongly suggest that if V is local and the scattering is rotationally invariant, then V is rotationally invariant.

In Sec. 3 it is shown that if nonlocal V's are considered, there exists a class of noninvariant interactions which yield an invariant $|\langle\mathbf{k}'|T|\mathbf{k}\rangle|^2$. However, as is shown in Sec. 4, the noninvariance of these interactions may be revealed by experiments in which two coherent beams are incident on the target. In this connection, it is proved that the requirement of physical invariance for arbitrary incident wave packets reduces to the requirement that the matrix elements of $T-\mathcal{R}^{-1}T\mathcal{R}$ (where $\mathcal{R}$ is the rotation operator) vanish on the energy shell. Note that if $\mathcal{R}^{-1}V\mathcal{R}=V$, we have $T-\mathcal{R}^{-1}T\mathcal{R}\equiv0$.

In Sec. 5 it is shown that there exist noninvariant V's which yield full physical invariance in scattering experiments. One example of this kind is obtained by using a trick suggested by the gauge invariance of electromagnetic interactions. It is noteworthy that the examples found here do not scale, i.e., if $V\to\lambda V$, the invariance is lost. We have not been able to find any examples which do scale and it may be that none exist. Note that such a situation is not completely unfamiliar: The gauge invariance of $H'=(\mathbf{p}-e\mathbf{A})^2/2m-\mathbf{p}^2/2m$ is lost if $H'\to\lambda H'$, $\lambda\neq1$. An approach to finding the class of potentials which do scale is indicated in the Appendix. Section 6 contains a discussion of the results.

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2. ROTATIONAL INVARIANCE AND LOCAL POTENTIALS

We denote the transition operator T, determined by the energy E and the interaction V, variously as

$$T=T(E)=T(E;\,V)=T_v\equiv V+V(E-H+i\epsilon)^{-1}V, \quad (2.1)$$

where $H=H_0+V$, $H_0=\mathbf{p}^2/2m$. (For the case of two-particle scattering, E is the c.m. energy and m is the reduced mass.) The standard plane waves are denoted by

$$|\mathbf{k}\rangle=e^{i\mathbf{k}\cdot\mathbf{r}},$$

and a plane wave with arbitrary phase is denoted by $\Phi_{\mathbf{k}}$, i.e.,

$$\Phi_{\mathbf{k}}=e^{i\varphi(\mathbf{k})}|\mathbf{k}\rangle,$$

with $\varphi(\mathbf{k})$ real but arbitrary. We also write

$$T_{\mathbf{k}',\mathbf{k}}=\langle\mathbf{k}'|T|\mathbf{k}\rangle$$

whether or not we are on the energy shell, i.e., whether or not $|\mathbf{k}|=|\mathbf{k}'|=(2mE)^{1/2}$.

The rotation operator corresponding to a rotation R (a 3×3 real orthogonal matrix) is denoted by $\mathcal{R}$, so that, in coordinate space,

$$\mathcal{R}\psi(\mathbf{x})=\psi(R^{-1}\mathbf{x}).$$

The interaction V is rotationally invariant if and only if

$$\mathcal{R}^{-1}V\mathcal{R}=V \quad (2.2)$$

for all R. For convenience of reference we now state the following as a theorem:

Theorem 1. $\mathcal{R}^{-1}V\mathcal{R}=V\Rightarrow T_{\mathbf{k}',\mathbf{k}}=T_{R\mathbf{k}',R\mathbf{k}}$ for all $\mathbf{k}'$, $\mathbf{k}$.

This theorem is indeed obvious and trivial since Eq. (2.1) and the hypothesis together with $\mathcal{R}^{-1}H_0\mathcal{R}=H_0=\mathbf{p}^2/2m$ imply that $\mathcal{R}^{-1}T\mathcal{R}=T$. The problem with which we are concerned is whether the implication also works in the other direction or if not, what then is implied by the statement that $T_{\mathbf{k}',\mathbf{k}}=T_{R\mathbf{k}',R\mathbf{k}}$, especially when $|\mathbf{k}'|=|\mathbf{k}|=(2mE)^{1/2}$.

Theorem 2. If $T(E)$ is rotationally invariant for some energy E, i.e., if $\mathcal{R}^{-1}T(E)\mathcal{R}=T(E)$ for some E, then V is also rotationally invariant.

This immediately follows from the relation[1]

$$V=T\left(1+\frac{1}{E-H_0+i\epsilon}T\right)^{-1}, \quad (2.3)$$

and from the fact that H_0 is rotationally invariant.

Equivalently, by virtue of the completeness of the states $|\mathbf{k}\rangle$, if $\langle\mathbf{k}'|T(E)|\mathbf{k}\rangle$ is rotationally invariant for some E and all $\mathbf{k}'$, $\mathbf{k}$, then V is also rotationally invariant. Note also that it then follows from Theorem 1 that $T(E)$ is rotationally invariant for all E.

We consider next the less strict condition that the T-matrix be rotationally invariant on the energy shell, i.e., for all $\mathbf{k}'$, $\mathbf{k}$ such that $|\mathbf{k}'|=|\mathbf{k}|=(2mE)^{1/2}$, and that nothing is known elsewhere.

Theorem 3(a). If V is local and if, for all E, $T_{\mathbf{k}',\mathbf{k}}(E;\lambda V)=T_{R\mathbf{k}',R\mathbf{k}}(E;\lambda V)$ on the energy shell and for all λ in some neighborhood of $\lambda=0$ in which $T_{\mathbf{k}',\mathbf{k}}$ is analytic in λ, then $\mathcal{R}^{-1}V\mathcal{R}=V$.

Proof. Since $T_{\mathbf{k}',\mathbf{k}}$ is analytic in this neighborhood, the perturbation expansion for $T_{\mathbf{k}',\mathbf{k}}$ is a valid expansion in powers of λ. We then require that the coefficient of each order of λ in the expansion be rotationally invariant.

We have

$$\langle\mathbf{k}'|T|\mathbf{k}\rangle=\langle R\mathbf{k}'|T|R\mathbf{k}\rangle,$$

with $|\mathbf{k}'|=|\mathbf{k}|$, so that

$$\langle\mathbf{k}'|\mathcal{R}^{-1}T\mathcal{R}-T|\mathbf{k}\rangle=0,$$

or

$$\int\int e^{-i\mathbf{k}'\cdot\mathbf{r}'}(\mathcal{R}^{-1}T\mathcal{R}-T)e^{i\mathbf{k}\cdot\mathbf{r}}d\mathbf{r}'d\mathbf{r}=0.$$

The first-order term in the expansion gives

$$\int\int e^{-i\mathbf{k}'\cdot\mathbf{r}'}[V(R\mathbf{r}',R\mathbf{r})-V(\mathbf{r}',\mathbf{r})]e^{i\mathbf{k}\cdot\mathbf{r}}d\mathbf{r}'d\mathbf{r}\big|_{|\mathbf{k}'|=|\mathbf{k}|}=0.$$

With $V(\mathbf{r}',\mathbf{r})=V(\mathbf{r})\delta(\mathbf{r}'-\mathbf{r})$, this reduces to

$$\int[V(R\mathbf{r})-V(\mathbf{r})]e^{i\mathbf{K}\cdot\mathbf{r}}d\mathbf{r}=0,$$

where $\mathbf{K}=\mathbf{k}-\mathbf{k}'$, with $|\mathbf{k}'|=|\mathbf{k}|$. Since the above must hold for all E, $\mathbf{K}$ can then assume *any* vector value by the appropriate choice of the unit vectors $\hat{k}$, $\hat{k}'$ and of E and so, by the uniqueness of the Fourier transform of a function, $V(R\mathbf{r})=V(\mathbf{r})$ or, equivalently, $\mathcal{R}^{-1}V\mathcal{R}=V$.

More generally we have the following theorem:

Theorem 3(b). If $V=\sum_{n=1}^{N}\lambda^nV_n$ and V is local and if perturbation theory is applicable [T an analytic function of λ as in Theorem 3(a)], then the rotational invariance of $T_{\mathbf{k}',\mathbf{k}}(E;\,V)$ implies the rotational invariance of V.

This follows by investigating the lowest orders in λ and using an inductive proof.

Corollary. If $V=\sum_{n=1}^{N}V_n$, V_n local, and if $V(\lambda)=\sum_{n=1}^{N}\lambda^nV_n$ is such that the hypothesis of Theorem 3(b) is valid in some neighborhood of $\lambda=0$, then V_n, $n=1,\cdots,N$, and hence $V=V(1)$, are rotationally invariant.

These results suggest, but do not prove, that if V is local, then the rotational invariance of the T matrix on the energy shell for all energies implies that V is rotationally invariant.

3. PHASE EQUIVALENT POTENTIALS

We now exhibit the following class, C_v, of potentials which yield a rotationally invariant $|T_{\mathbf{k}',\mathbf{k}}|$ but which are, in general, rotationally noninvariant.

[1] We assume that the required inverse exists for some E for which $T(E)$ is rotationally invariant.

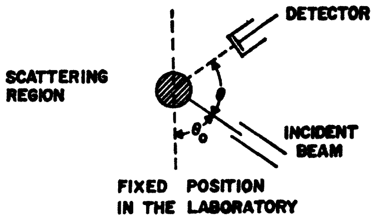

FIG. 1. Scattering experiment of type E.

Definition. The potential V belongs to the class C_v if and only if

$$V = e^{-i\alpha(\mathbf{p})} V_i e^{i\alpha(\mathbf{p})}, \tag{3.1}$$

where V_i is rotationally invariant and where $\alpha(\mathbf{u})$ is an arbitrary real valued function when $\mathbf{u}$ is a real vector. We shall also say that V is phase equivalent to V_i.

Note that unless $\alpha(\mathbf{u})$ is a function of $\mathbf{u}^2$ only, V is not rotationally invariant.

Theorem 4. If V belongs to the class C_v, then $|T_{\mathbf{k}',\mathbf{k}}|$ is rotationally invariant for all $\mathbf{k}$, $\mathbf{k}'$.

Proof. Since

$$[\mathbf{p},H_0]=0,$$

Eq. (2.1) implies that

$$T = e^{-i\alpha(\mathbf{p})} T_i e^{i\alpha(\mathbf{p})}, \tag{3.2}$$

where the operator

$$T_i = V_i + V_i \frac{1}{E - H_0 - V_i + i\epsilon} V_i \tag{3.3}$$

is a rotationally invariant operator. Hence,

$$\langle \mathbf{k} | T | \mathbf{k} \rangle = e^{-i[\alpha(\mathbf{k}')-\alpha(\mathbf{k})]} \langle \mathbf{k}' | T_i | \mathbf{k} \rangle, \tag{3.4}$$

and so

$$|T_{\mathbf{k}',\mathbf{k}}| = |(T_i)_{\mathbf{k}',\mathbf{k}}|.$$

In the next section, we look further into the physical aspects of the potentials of the class C_v. Here we note only that C_v includes as a subclass potentials which are rotationally invariant when considered with regard to rotations about a point with vector coordinate $\mathbf{a}$ with respect to the point left fixed by the rotations R. This is most easily seen by taking $\alpha(\mathbf{u}) = \mathbf{a}\cdot\mathbf{u}$, and $V_i = V_i(\mathbf{r})$ a local potential, from which Eq. (3.1) reduces to $V = V_i(\mathbf{r}-\mathbf{a})$. Clearly an incident infinite plane wave cannot be used to measure the "location" of V, but an incident wave packet can. The generalization of this remark is exploited in the next section.

4. COHERENT BEAMS AND ROTATIONAL INVARIANCE

Consider the scattering experiment described by Fig. 1. We shall denote this as experiment E. Let $I(\theta,\theta_0)$ denote the intensity detected. Then if the scattering is rotationally invariant, $I(\theta,\theta_0)$ is independent of θ_0. We say then that the experiment E is positive.

Consider a potential V belonging to C_v, Eq. (3.1), and the corresponding T operator, T_v. Then, omitting kinematical factors,

$$I(\theta,\theta_0) = |\langle \mathbf{k}' | T_v | \mathbf{k} \rangle|^2.$$

Thus, by Theorem 4, $I(\theta,\theta_0)$ is rotationally invariant, i.e., independent of θ_0. We now see that if a V_i gives rise to $I(\theta,\theta_0)$, then any V which is phase equivalent to V_i gives rise to the same scattering. Thus even if $I(\theta,\theta_0)$, is observed to be independent of θ_0, V is not necessarily rotationally invariant.

Now consider the experiment described by Fig. 2. We shall call this experiment E'. Here the incident beam has been split into two parts so that two *coherent* beams are incident upon the scattering center. We denote by $I'(\theta,\theta_0)$ the new intensity at the detector.

Now if we were given that $I(\theta,\theta_0)$ of experiment E is independent of θ_0, we might hazard a guess that $I'(\theta,\theta_0)$ of experiment E' is also independent of θ_0. We would like here to emphasize that, in general, this is *not* the case.

We again consider a potential V of the class C_v. Then in the experiment E', the incident state Φ_i has the form

$$\Phi_i = c_1|\mathbf{k}_1\rangle + c_2|\mathbf{k}_2\rangle.$$

Hence

$$\begin{aligned} I'(\theta,\theta_0) = {} & |c_1|^2 |F(\mathbf{k}',\mathbf{k}_1)|^2 + |c_2|^2 |F(\mathbf{k}',\mathbf{k}_2)|^2 \\ & + 2\,\mathrm{Re}\, c_1^* c_2 \langle \mathbf{k}' | T_v | \mathbf{k}_1 \rangle^* \langle \mathbf{k}' | T_v | \mathbf{k}_2 \rangle, \end{aligned} \tag{4.1}$$

where

$$F(\mathbf{k}',\mathbf{k}) = \langle \mathbf{k}' | T_i | \mathbf{k} \rangle,$$

and T_i is defined by Eq. (3.3). The expression (4.1) depends on θ_0 since $c_1^* c_2 \neq 0$ and the coefficient of $c_1^* c_2$ is

$$F^*(\mathbf{k}',\mathbf{k}_1) F(\mathbf{k}',\mathbf{k}_2) e^{i[\alpha(\mathbf{k}_2)-\alpha(\mathbf{k}_1)]},$$

which, in general, depends on θ_0 if $\alpha(\mathbf{u})$ is not rotationally invariant.

Thus, if experiment E' is positive, i.e., $I'(\theta,\theta_0)$ is independent of θ_0 for any given value of c_1, c_2, φ_1, and φ_2, and if the potential V belongs to the class C_v, we are led to the conclusion that V is rotationally invariant [i.e., $\alpha(\mathbf{u}) = \alpha(\mathbf{u}^2)$ only]. We can, of course, say that if E' is positive, then E, being a special case of E', will also be positive. We now see, however, that the converse is not generally true. The above description (see Figs. 1 and 2) and notation are those appropriate for 2 dimen-

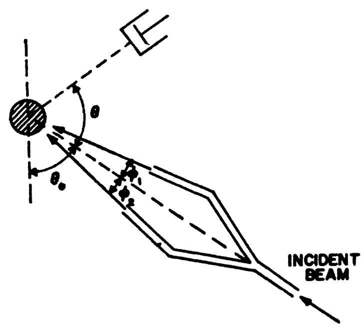

FIG. 2. Scattering experiment of type E'.

sions. It is clear that the whole argument is also valid in 3 dimensions.

Thus if experiment E is positive, $|\langle\Phi_{\mathbf{k}'}|T|\Phi_{\mathbf{k}}\rangle| = |\langle\Phi_{\mathbf{k}'}|\mathcal{R}^{-1}T\mathcal{R}|\Phi_{\mathbf{k}}\rangle|$ on the energy shell and the absolute value signs may not, in general, be removed, as Eq. (3.4) shows. We now show that if E' is positive (and hence E), we can conclude that, on the energy shell, $\langle\Phi_{\mathbf{k}'}|T|\Phi_{\mathbf{k}}\rangle = \langle\Phi_{\mathbf{k}'}|\mathcal{R}^{-1}T\mathcal{R}|\Phi_{\mathbf{k}}\rangle$.[2]

Theorem 5. If $|c_1\langle\Phi_{\mathbf{k}'}|T|\Phi_{\mathbf{k}_1}\rangle + c_2\langle\Phi_{\mathbf{k}'}|T|\Phi_{\mathbf{k}_2}\rangle|^2$ is rotationally invariant for all c_1, c_2 and for all $\mathbf{k}'$, $\mathbf{k}_1$, $\mathbf{k}_2$, such that $|\mathbf{k}'| = |\mathbf{k}_1| = |\mathbf{k}_2| = (2mE)^{1/2}$, then

$$\langle\Phi_{\mathbf{k}'}|T-\mathcal{R}^{-1}T\mathcal{R}|\Phi_{\mathbf{k}}\rangle = 0 \tag{4.2}$$

with $|\mathbf{k}'| = |\mathbf{k}| = (2mE)^{1/2}$.

Proof. By choosing $c_1{}^*c_2$ to be first real and then imaginary in the expression corresponding to Eq. (4.1), we get that

$$\langle\Phi_{\mathbf{k}'}|T|\Phi_{\mathbf{k}_1}\rangle^*\langle\Phi_{\mathbf{k}'}|T|\Phi_{\mathbf{k}_2}\rangle = \langle\mathcal{R}\Phi_{\mathbf{k}'}|T|\mathcal{R}\Phi_{\mathbf{k}_1}\rangle^*\langle\mathcal{R}\Phi_{\mathbf{k}'}|T|\mathcal{R}\Phi_{\mathbf{k}_2}\rangle. \tag{4.3}$$

We now write

$$\langle\Phi_{\mathbf{k}'}|T|\Phi_{\mathbf{k}}\rangle = |\langle\Phi_{\mathbf{k}'}|T|\Phi_{\mathbf{k}}\rangle|e^{i\beta(\mathbf{k}',\mathbf{k})}$$

and

$$\langle\mathcal{R}\Phi_{\mathbf{k}'}|T|\mathcal{R}\Phi_{\mathbf{k}}\rangle = |\langle\mathcal{R}\Phi_{\mathbf{k}'}|T|\mathcal{R}\Phi_{\mathbf{k}}\rangle|e^{i\beta_R(\mathbf{k}',\mathbf{k})}.$$

Since

$$|\langle\Phi_{\mathbf{k}'}|T|\Phi_{\mathbf{k}}\rangle| = |\langle\mathcal{R}\Phi_{\mathbf{k}'}|T|\mathcal{R}\Phi_{\mathbf{k}}\rangle|,$$

Eq. (4.3) implies that

$$\beta(\mathbf{k}',\mathbf{k}_1)-\beta_R(\mathbf{k}',\mathbf{k}_1) = \beta(\mathbf{k}',\mathbf{k}_2)-\beta_R(\mathbf{k}',\mathbf{k}_2).$$

But $\hat{k}_1$ and $\hat{k}_2$ are independent and hence $\beta(\mathbf{k}',\mathbf{k}) - \beta_R(\mathbf{k}',\mathbf{k}) = \gamma_R(\mathbf{k}')$, a function independent of $\mathbf{k}$.

Now unitarity implies that the total cross section, σ_T, for an incident beam of momentum $\mathbf{k}'$ is proportioned to $\sin\beta(\mathbf{k}',\mathbf{k}')$ (the optical theorem). Thus, the rotational invariance of σ_T implies that

$$\beta(\mathbf{k}',\mathbf{k}') = \beta_R(\mathbf{k}',\mathbf{k}')$$

for all R and $\mathbf{k}'$, so that

$$\gamma_R(\mathbf{k}') = 0$$

or

$$\beta(\mathbf{k}',\mathbf{k}) = \beta_R(\mathbf{k}',\mathbf{k})$$

for all R, $\mathbf{k}'$, and $\mathbf{k}$. It follows that

$$\langle\Phi_{\mathbf{k}'}|T|\Phi_{\mathbf{k}}\rangle = \langle\mathcal{R}\Phi_{\mathbf{k}'}|T|\mathcal{R}\Phi_{\mathbf{k}}\rangle,$$

which is equivalent to Eq. (4.2).

It is now of interest to see whether or not there exist any rotationally noninvariant potentials giving positive results for experiment E' and hence such that Eq. (4.2) is satisfied although $T-\mathcal{R}^{-1}T\mathcal{R}\neq 0$. In Sec. 5, we give two examples, one of which is discussed in some detail.

[2] This result depends in an essential way on the fact that T is a linear operator. The power of the linearity of T was stressed to us by Dr. R. Stora.

5. EXAMPLES

Consider a rotationally invariant interaction generated from $H_0(\mathbf{p}) = \mathbf{p}^2/2m$ by $\mathbf{p}\to\mathbf{p}-\lambda\mathbf{A}$ and where $\mathbf{A} = A(r)\hat{r}$. This interaction, as far as the following discussion is concerned, does not necessarily have anything to do with an electromagnetic interaction although the discussion is evidently guided by the idea of gauge invariance in electrodynamics. Hence, we are not concerned here with the question of the existence of physical currents which would generate such an $\mathbf{A}$.

There are then scattering eigenstates ψ for the total Hamiltonian $H = H_0(\mathbf{p}-\lambda\mathbf{A})$ such that the asymptotic form of ψ is given by

$$\psi \to e^{i\mathbf{k}\cdot\mathbf{r}} + f(\theta,\varphi)e^{ikr}/r,$$

provided $\mathbf{A}\to 0$ sufficiently rapidly as $r\to\infty$. Because of the form of $\mathbf{A}$, $f(\theta,\varphi)$ and $T_{\mathbf{k}',\mathbf{k}}$ are rotationally invariant.

Let us now perform a gauge transformation on the "vector potential" $\mathbf{A}$:

$$\mathbf{A}\to\mathbf{A}' = \mathbf{A}+\nabla\chi, \tag{5.1}$$

where χ is an arbitrary function. Then we know that

$$\psi' = e^{i\lambda\chi}\psi$$

is a solution of the Schrödinger equation with the new total Hamiltonian $H' = H|_{\mathbf{A}\to\mathbf{A}'}$. Thus, if we also have that $\chi\to 0$ as $r\to\infty$, the asymptotic form of ψ' is the same as that of ψ.

Now if χ is not rotationally invariant, H_I', the interaction part of H', will also not be rotationally invariant. But, since the asymptotic form of ψ' and ψ are the same, H' induces the same scattering as H and the resulting $T'_{\mathbf{k}',\mathbf{k}}$ is rotationally invariant, at least, on the energy shell. The possibility of this state of affairs is related to the fact that

$$T'_{\mathbf{k}',\mathbf{k}} = T'_{R\mathbf{k}',R\mathbf{k}}$$

holds only *on the energy shell*, and off the energy shell the equality is no longer true.

Let us now look at this more quantitatively from the point of view of perturbation theory. We have

$$H_0 = \mathbf{p}^2/2m,$$
$$H = (\mathbf{p}-\lambda\mathbf{A})^2/2m = H_0 + H_I(\mathbf{A}),$$
$$H_I(\mathbf{A}) = -(\lambda/2m)(\mathbf{p}\cdot\mathbf{A}+\mathbf{A}\cdot\mathbf{p}) + (\lambda^2/2m)\mathbf{A}^2.$$

We make the gauge transformation (5.1) and obtain the new interaction

$$H_I' = H_I(\mathbf{A}') = H_I(\mathbf{A}) + h_I(\chi,\mathbf{A}). \tag{5.2}$$

h_I can be brought into the following form:

$$h_I(\chi,\mathbf{A}) = g^{(1)}(\chi,\mathbf{A}) + g^{(2)}(\chi,\mathbf{A}),$$

where

$$g^{(1)}(\chi,\mathbf{A}) = -2i\lambda[H_0,\chi],$$

and

$$g^{(2)}(\chi,\mathbf{A}) = (\lambda^2/2m)\{2i\mathbf{A}-[\mathbf{p},\chi]\}\cdot[\mathbf{p},\chi].$$

That perturbation theory is valid means that the T matrix is an analytic function of a "coupling constant" c in some neighborhood of $c=0$. Here our "coupling constant" is the "charge" λ. Thus, the perturbation expansion is really an expansion in powers of λ. It is then expected that the part involving χ of *each order in* λ vanishes by itself when the initial and final states are put on the energy shell. Since H_I' involves more than one power of λ, we must be careful to gather all the χ terms pertaining to the order of λ in which we are interested in order to see this vanishing.

We consider the first term, $\langle\mathbf{k}'|H_I'|\mathbf{k}\rangle$, of the perturbation expansion for $T'_{\mathbf{k}',\mathbf{k}}$. This term gives, as far as χ is concerned,

$$\langle\mathbf{k}'|h_I(\chi,\mathbf{A})|\mathbf{k}\rangle = g^{(1)}{}_{\mathbf{k}',\mathbf{k}}+g^{(2)}{}_{\mathbf{k}',\mathbf{k}},$$

where

$$g^{(1)}{}_{\mathbf{k}',\mathbf{k}}=\langle\mathbf{k}'|g^{(1)}(\chi,\mathbf{A})|\mathbf{k}\rangle \text{ and } g^{(2)}{}_{\mathbf{k}',\mathbf{k}}=\langle\mathbf{k}'|g^{(2)}(\chi,\mathbf{A})|\mathbf{k}\rangle.$$

It can be seen that $g^{(1)}{}_{\mathbf{k}',\mathbf{k}}$ does not, in general, vanish. But it does vanish on the energy shell. On the other hand, $g^{(2)}{}_{\mathbf{k}',\mathbf{k}}$ is of order λ^2 and does not generally vanish even when on the energy shell. In fact, when on the energy shell, $g^{(2)}{}_{\mathbf{k}',\mathbf{k}}$ is canceled by the λ^2 part involving χ of the term $\langle\mathbf{k}'|H_I'G_0(E)H_I'|\mathbf{k}\rangle$. This illustrates our statements in the previous paragraph.

An interesting point in this mechanism of the vanishing of the χ terms is the following. It may be thought that when all the χ terms in order λ^2 are brought together we could re-arrange them and rewrite them in such a way that they are equivalently expressed in the form

$$\langle\mathbf{k}'|[H_0,\rho_E(\chi,\mathbf{A})]|\mathbf{k}\rangle \tag{5.3}$$

when vanishing on the energy shell is quickly seen. But this is not so. An explicit calculation shows that the vanishing of the χ terms takes place only when we put the energies of the initial and final states equal to E whereas expression (5.3) requires for vanishing only that the energies of the initial and final states be equal. Only the term in order λ which does not involve $G_0(E)$ can be brought into the form of (5.3).

Another example may be constructed as follows. Let V be a rotationally invariant potential and have bound states Ψ_α. Now construct a new potential V' by putting

$$V'=V+\sum_{\alpha,\beta=1}^{N}\Gamma_{\alpha\beta}|\Psi_\alpha\rangle\langle\Psi_\beta|. \tag{5.4}$$

Then the continuum eigenstates of the Schrödinger equation with the potential V also satisfy the Schrödinger equation with the potential V'. Thus V' and V must give rise to the same scattering. But since Ψ_α is, in general, not rotationally invariant, we can, in fact, choose $\Gamma_{\alpha\beta}$ to be simply c numbers and such that V' is noninvariant. Hence, V' is a rotationally noninvariant potential which gives rise to rotationally invariant scattering.

6. DISCUSSION

The results of the previous sections show that the interactions under consideration may be put into two classes. Class C_{I} consists of those V for which experiment E' is positive, while class C_{II} contains those V for which only experiment E is positive. For $V\in C_{\mathrm{I}}$, $T_{\mathbf{k}',\mathbf{k}}$ is a rotational invariant on the energy shell while for $V\in C_{\mathrm{II}}$, $|T_{\mathbf{k}',\mathbf{k}}|$ is invariant, but $T_{\mathbf{k}',\mathbf{k}}$ is not. The noninvariant V's $\in C_v$ [see Eq. (3.1)] belong to C_{II}. The interactions defined by Eqs. (5.2) and (5.4) belong to C_{I}, which, of course, also includes the class of rotationally invariant V.

The existence of experiments of the type E' indicates that class C_{II} is at most of mathematical interest. Thus, we confine ourselves now to class C_{I}. A necessary and sufficient condition for V to belong to C_{I} (see Theorem 5) is that $\langle\mathbf{k}|T-\mathcal{R}^{-1}T\mathcal{R}|\mathbf{k}\rangle$ or, equivalently, $\langle\mathbf{k}'|[\mathbf{l},T]|\mathbf{k}\rangle$ vanish on the energy shell ($\mathbf{l}=\mathbf{r}\times\mathbf{p}$). We have not been able to find the general form of V which satisfies this condition. For the case where a power series expansion in V is possible, the problem may be "mechanized" in a manner indicated in the Appendix. The resulting infinite set of coupled integral equations is not very transparent. It is possible that no nontrivial solution to these equations exists. (See the remarks in the introduction on scaling of the interaction.[3])

A relevant question which may be raised at this point is the following. Given that the scattering is rotationally invariant, can one assume without loss of generality that the interaction which induces this scattering is rotationally invariant? For each of the non invariant interactions given previously there exists an invariant interaction which yields the same scattering. If this is true generally (we have *not* determined this) then the assumption that $\mathcal{R}^{-1}V\mathcal{R}=V$ may be made without loss of generality, as far as an analysis of an experiment of the type E' is concerned. We note that if V_n and V_i are, respectively, noninvariant and invariant interactions which yield the same scattering then the question of whether the interaction is "really" V_n rather than V_i may be resolved by performing a scattering experiment in which the interaction is $V+U$, where U is an invariant interaction, and seeing whether the scattering is still rotationally invariant, *provided* such an experiment is physically realizable.[4] Thus, again, the greater the class of "in principle" realizable experiments for testing an invariance the more limited the possible forms of interaction, as was also seen in Sec. 4. This is in accord with the remark of Wigner that "the fewer the experiments that are permissible 'in principle,' the

[3] If we consider the Dirac equation in relativistic quantum mechanics, we can easily construct a noninvariant interaction which does scale and which gives rise to rotationally invariant scattering by using the gauge trick of Sec. 5, i.e., by putting $H'=-\lambda\boldsymbol{\alpha}\cdot(\mathbf{A}+\boldsymbol{\nabla}\chi)$.

[4] This point was brought out in a conversation with Dr. E. C. G. Sudarshan.

easier it will be to satisfy the principle of invariance, the less meaningful such a principle will become."[5]

In conclusion, we believe that the above serves to illustrate, in an explicit manner, that it is easy to build too much invariance into a theory when the observable quantities in the theory are rather indirectly related to the elements in terms of which the theory is formulated. It is conceivable that a theory which is in fact physically relevant might be rejected in this way.

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APPENDIX

We consider here the problem for potentials of the form λV, where λ is the coupling constant, and, for simplicity, we restrict ourselves to scattering in 2 dimensions. We also assume that perturbation theory is valid, i.e., that the T matrix is analytic in λ in some neighborhood of $\lambda=0$. The fact that the interaction is linear in λ means that we can demand that each term, $\langle\mathbf{k}'|H_IG_0(E)\cdots H_I|\mathbf{k}\rangle$, is separately rotationally invariant when $|\mathbf{k}'|=|\mathbf{k}|=(2mE)^{1/2}$. The equations below are the results of the computation.

We first expand the general nonlocal potential $\lambda V(\mathbf{r}',\mathbf{r})$ as follows:

$$V(\mathbf{r}',\mathbf{r})=\sum_{m,m'=-\infty}^{\infty} v_{m',m}(r',r)e^{-im'\theta'}e^{im\theta}$$

and put

$$v_{m',m}(r',r)=\int\int_0^{\infty} \nu_{m',m}(K',K)J_{m'}(K'r')J_m(Kr)dK'dK,$$

where $J_m(z)$ is the usual Bessel function of order m. We then obtain the following set of coupled equations as the condition that the T matrix be rotationally invariant on the energy shell:

$$\nu_{n+a,n}(\mathbf{k},\mathbf{k})=0$$

and, in general, the $(j+1)$th term, $j=1, 2, \cdots$, of the perturbation expansion gives

$$\sum_{l_1,\cdots,l_j=-\infty}^{\infty}\int\cdots\int_0^{\infty} K_1dK_1\cdots \times K_jdK_j\nu_{n+a,l_j}(k,K_j)\frac{1}{K_j^2-k^2-i\epsilon} \times\nu_{l_j,l_{j-1}}(K_j,K_{j-1})\cdots\frac{1}{K_1^2-k^2-i\epsilon}\nu_{l,n}(K_1,k)=0, \quad (7.1)$$

for all k, n and for all $a\neq 0$.

[5] E. P. Wigner, Nuovo Cimento 3, 517 (1956).